# The Signature Function and Higher-Dimensional Objects

Conrad sen Kyne

**Abstract**

The signature function is discussed in more depth. We view the consequences of some definitions of matrices. A right near-ring is discovered which is closely related to the signature near-ring. Finally, I show a generalization of the signature function to 3D matrices, and discover some very smooth formulas for several one-beginning cubes. $N$-dimensional matrices are defined, with a corresponding generalization of the signature function. Finally, various constructions are examined.

## Contents

**1 The signature function and matrices** **2**
- 1.1 Matrix operations . . . 2
- 1.2 The signature function over matrices . . . 2
- 1.3 Power triangles . . . 2
- 1.4 The power triangle monoid . . . 3
- 1.5 Another near-ring . . . 4

**2 The signature function in higher dimensions** **5**
- 2.1 Canonical one-beginning objects . . . 5
- 2.2 The power triangular prism . . . 6
- 2.3 The binomial simplex . . . 7
- 2.4 G-prisms . . . 7
- 2.5 Arbitrary one-beginning objects . . . 8
- 2.6 Higher-dimensional signature convolution . . . 8

**3 Conclusion** **9**

# 1 The signature function and matrices

## 1.1 Matrix operations

Before we explore matrices in more detail, it is important to define the matrix operations which we will be using.

The set of matrices forms a group under the following abelian addition operation:

$$(M+N)(n,y) = M(n,y) + N(n,y)$$

The set of matrices also forms a monoid under the following multiplication operation:

$$MN(n,y) = \sum_{k=0}^{\infty} M(n,k) \cdot N(k,y)$$

This operation is slightly different than traditional matrix multiplication, in that it does not have dimensional constraints. We assume that each matrix is padded with zeroes, so as to make either addition or multiplication possible.

Sequences may be considered as scalars for the purpose of multiplication by matrices. This multiplication along with matrix addition results in a bimodule.

$$dM(n,y) = \sum_{k=0}^{y} d_k \cdot M(n, y-k)$$

## 1.2 The signature function over matrices

The signature function over a matrix is performed via antidiagonal summation. Because of how varied matrices can be, the result of the signature function on a matrix will not necessarily begin with 1.

$$F_M(n) = \sum_{k=0}^{n} M(n-k, k)$$

Scalar multiplication is linear over $F$ with respect to matrix addition.

$$F_{dM+gN} = dF_M + gF_N$$

## 1.3 Power triangles

The power triangles are a subset of matrices with some interesting properties. They are constructed with a sequence $d$ like so:

$$T_d(n,y) = d_y^n$$

These matrices are "one-beginning", meaning $T_d(0,0) = 1$ so the signature function always yields a one-beginning sequence. Power triangles are the canonical one-beginning matrices, and have the following identity:

$$F_{T_d} = F_d$$

The following shorthand will be used to simplify future notation:

$$F^{-1}(F_{T_d}) \to F_{T_d}^{-1}$$

T is closed under matrix multiplication. Multiplication of power triangles also yields interesting signatures:

$$F_{T_a \otimes T_b}^{-1} = \sum_{k=0}^{\infty} a_k b^k$$

Power triangles have a very interesting relationship to signature addition through subtraction and **scalar division**, which is defined as:

$$\frac{T_d}{g}(n) = \frac{d^n}{g}$$

Then we arrive at signature addition in an interesting way:

$$\sum_{k=0}^{n} \frac{T_d - T_g}{d - g}(k+1, n-k) = F_{d \oplus g}(n)$$

Note that when $d = g$ exactly, this algorithm yields $0/0$, but the limit as $g$ approaches $d$ is still $F_{d \oplus g}$. The closed form using sequences instead of matrices is very similar:

$$\sum_{k=0}^{n} \frac{d^{k+1} - g^{k+1}}{d - g}(n-k) = F_{d \oplus g}(n)$$

Signature convolution is constructed by convolving the antidiagonal with the signature function of $g$:

$$\sum_{k=0}^{n} T_d(n-k, k) \cdot F_g(n-k) = F_{d \circ g}(n)$$

## 1.4 The power triangle monoid

$(T, \otimes)$ is closed (as is the set of one-beginning matrices) and produces an interesting homomorphism:

$$T_a \otimes T_b = T_{a*b}$$

Where the operation $*$ is defined as:

$$a * b = \sum_{k=0}^{\infty} a_k b^k$$

This associative operation forms a monoid with identity element $x$. There are also some interesting results when multiplying by $x^n$:

$$x^n * d = d^n$$

$$d * x^n = A_d^n$$

Where $A$ is the aeration function.

## 1.5 Another near-ring

Together with sequence addition, the power triangle monoid forms a right near-ring as multiplication distributes over addition on the right.

$$\begin{aligned}
(a+b) * c &= \sum_{k=0}^{\infty} (a+b)_k c^k \\
&= \sum_{k=0}^{\infty} (a_k + b_k) c^k \\
&= \sum_{k=0}^{\infty} a_k c^k + b_k c^k \\
&= \sum_{k=0}^{\infty} a_k c^k + \sum_{k=0}^{\infty} b_k c^k \\
&= a * c + b * c
\end{aligned}$$

When the left operand is a number (ie a single-digit sequence) then it absorbs:

$$n * a = n$$

This includes 0, which also only absorbs on the left. This is a simple consequence of the definition of multiplication:

$$n * a = \sum_{k=0}^{\infty} n_k a^k = n \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + ... = n$$

(The fact that 0 does not absorb on the right may be viewed as a side effect of the implicit definition of $0^0 = 1$.)

Due to left absorption, every number is also idempotent:

$$n * n = n$$

This operation shares an interesting connection to signature multiplication through the following identity:

$$a * bx = \sum_{k=0}^{\infty} a_k (bx)^k = \frac{b \circ a}{b}$$

There is a notable exception to this formula, and that is when $b = 0$:

$$\frac{0 \circ a}{0}$$

Now here's where things get funky; Because we have previously implied that $0^0 = 1$, we are also implying that $\frac{0}{0} = 1$. This creates a rather funny cancellation:

$$\frac{0 \circ a}{0} = \frac{0 \cdot \sum_{k=0}^{\infty} 0^k x^k a_k}{0} = \sum_{k=0}^{\infty} 0^k x^k a_k = a_0$$

This creates an interesting relationship between the signature near-ring and this new right near-ring:

$$\sum_{k=0}^{n} \frac{a^{k+1} - b^{k+1}}{a - b}(n - k) = F_{a \oplus b}(n)$$

$$a * bx = \frac{b \circ a}{b}$$

# 2 The signature function in higher dimensions

## 2.1 Canonical one-beginning objects

While the signature function has been performed on both signatures and matrices, it can also be performed on 3D matrices. Take, for example, the following construction:

$$P_d(r_0, r_1, r_2) = d_{r_2}^{r_0 + r_1}$$

This is the canonical one-beginning object in three dimensions, a power cube. Then the signature function defined over such a cube is given by:

$$F_{P_d}(n) = \sum_{r_0 + r_1 + r_2 = n} P_d(r_0, r_1, r_2) = F_{d \oplus d}(n)$$

We can thus generalize the notion of a one-beginning object to $N$ dimensions and perform the signature function on it with relative ease:

$$P_d^N(r_0, r_1, ..., r_{N-1}) = d_{r_{N-1}}^{r_0 + r_1 + ... + r_{N-2}}$$

$$F_{P_d^N}(n) = \sum_{r_0+r_1+...+r_{N-1}=n} P_d^N(r_0, r_1, ..., r_{N-1}) = F_{d^{(N-1)}}(n)$$

Where $d^{(k)}$ is iterated signature addition given by:

$$d^{(k)} = \underbrace{d \oplus d \oplus ... \oplus d}_{k\ times}$$

We may also iterate signature subtraction:

$$d^{(-n)} = -d^{(n)} F_{d^{(n)}}$$

## 2.2 The power triangular prism

Let $W_d^3$ be given by the following function:

$$W_d^3(n, y, t) = d_t^y$$

We call this a **prism** because of the resultant shape of the matrix when each $n$-th element of $W_d^3$ corresponds to the same triangle. Then the following identity holds:

$$F_{W_d^3} = F_{d \oplus 1}$$

We may generalize this structure to N dimensions:

$$W_d^N(r_0, ..., r_{N-1}) = d_{r_{N-1}}^{r_{N-2}}$$

Then the signature function gives:

$$F_{W_d^N} = F_{d \oplus 1^{(N-2)}}$$

## 2.3 The binomial simplex

We can construct a generalization of Pascal's triangle in three dimensions:

$$C^3(r_0, r_1, r_2) = \binom{r_0}{r_1}\binom{r_0 - r_1}{r_2}$$

Then we have quite simply:

$$F_{C^3} = F_{1,2}$$

Noting that $\binom{n}{k} = \{1, 1\}_k^n$, we can substitute any set of signatures $d = [d_0, d_1, ...]$ each of which are two digits in length, and define $C^N$ as:

$$C^N(r_0, r_1, ..., r_{N-1}) = d_0^{r_0}(r_1)d_1^{r_1}(r_2)...d_{N-2}^{r_{N-3}}(r_{N-2})d_{N-1}^{r_{N-3}-r_{N-2}}(r_{N-1})$$

Then the signature of the resultant prism is:

$$\begin{aligned} F_{C^N}^{-1} = &\sum_{k=0}^{N-3} x^k \cdot d_{k,0} \cdot \prod_{t=0}^{k-1} d_{t,1} \\ &+ x^{N-2} \cdot d_{N-2,0} \cdot d_{N-1,0} \cdot \prod_{k=0}^{N-3} d_{k,1} \\ &+ x^{N-1} \cdot (d_{N-2,0} \cdot d_{N-1,1} + d_{N-2,1}) \cdot \prod_{k=0}^{N-3} d_{k,1} \end{aligned}$$

## 2.4 G-prisms

Let $d$ and $g$ be sets a signatures, with $N = |d|$ and $K = |g|$. Then let $g_n^r$ be defined as follows:

$$g_n^r = \sum_{k=0}^{r} F_{g_n}(k) \cdot x^k$$

Next, define $G$ as follows:

$$G(r_0, r_1, ..., r_{N+K-1}) = \left(g_0^{r_0} g_1^{r_1} ... g_{K-1}^{r_{K-1}} d_0^{r_K} d_1^{r_{K+1}} ... d_{N-1}^{r_{N+K-2}}\right)_{N+K-1}$$

Then the signature of this prism is given by:

$$F_G^{-1} = \bigoplus_{n=0}^{N-1} d_n \oplus \bigoplus_{k=0}^{K-1} (1 \oplus x A_{g_n}^2)$$

## 2.5 Arbitrary one-beginning objects

First, we can construct a function which takes two matrices, namely two power triangles, as arguments:

$$T \times M(r_0, r_1, r_2) = (T(r_0) \otimes M(r_1))(r_2)$$

In this case $M(n)$ is the $n$-th row of $M$. Then we have quite simply:

$$F_{T\times M} = F_T \otimes F_M$$

Now let's assume arbitrary dimensions; $N, K > 2$ for $T$ and $M$ respectively:

$$T{\times}M(r_0, r_1, ...r_{N+K-2}) = (T(r_0, r_1, ..., r_{N-2}) \otimes M(r_{N-1}, r_N, ..., r_{N+K-3}))\,(r_{N+K-2})$$

This is the **prismatic product** of $T$ and $M$. Then, identically to the two-dimensional case, we have:

$$F_{T\times M} = F_T \otimes F_M$$

If T and M are one-beginning, we also have:

$$F^{-1}_{T\times M} = F^{-1}_T \oplus F^{-1}_M$$

## 2.6 Higher-dimensional signature convolution

Suppose we have the the following prism:

$$T = T_{g_0} \times ... \times T_{g_{N-2}}$$

Then we can define a product over the general signature function:

$$\sum_{r_0+...+r_{N-1}=n} T(r_0, ..., r_{N-1}) \cdot F_s(r_0) = F_{T\circ s}(n)$$

Much like in two dimensions, we are multiplying the specific element of the product by the $r_0$-th element of $F_s$. But, we can take this a step further. Suppose we have instead $N-2$ signatures in $S$. Then the summation is instead:

$$\sum_{r_0+...+r_{N-1}=n} T(r_0, ..., r_{N-1}) \cdot \prod_{k=0}^{N-2} F_{S_k}(r_k) = F_{T\circ S}(n)$$

Then signature convolution gives the following identity:

$$T \circ S = \bigoplus_{k=0}^{N-2} g_k \circ S_k$$

This is the **signature dot product**. It reduces the signature function over an arbitrary-dimensional prismatic object to the signature sum of signature convolutions. This means a prismatic object does not need to be constructed for its signature to be computed; and conversely, every signature may be seen as representative of the traversal of an object of at least two dimensions.

## 3 Conclusion

There is plenty to explore of the signature function in higher dimensions. Repeatedly, we see the natural appearance of signature addition in the one-beginning forms shown. We also see the appearance of iterated signature addition, something not seen at all in exploration of one-beginning sequences. This additional structure is promising, as it makes signature addition a more fundamental consequent of the signature function.

The operations that emerge from these higher-dimensional objects are not the end. The next step forward is to search for a continuous generalization of the signature function. If such a function does exists, then I predict there will also be a canonical one-beginning continuous object, and that signature addition will emerge once again.